Holographic View of Cosmological Perturbations

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Abstract. The cosmological perturbation theory is revisited from the holographic point of view. In the case of the single brane model, it turns out that the AdS/CFT correspondence plays an important role. In the case of the two-brane model, it is shown that the effective equations of motion becomes the quasi-scalar-tensor gravity. It is also demonstrated that the radion anisotropy gives the CMB fluctuations through the Sachs-Wolfe effect.

Keywords: holographic, brane, cosmological perturbations

1. Introduction

In this paper, we would like to propose a new approach to the cosmology in the context of the warped compactification modeled by the action (Randall and Sundrum, 1999a; Randall and Sundrum, 1999b)

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left(\mathcal{R}+\frac{12}{l^2}\right) - \sigma\int d^4x\sqrt{-h} + \int d^4x\sqrt{-h}\mathcal{L}_{\rm matter} \ , \tag{1}$$

where σ, l, κ^2 are the brane tension, AdS curvature scale, and the gravitational constant, respectively. Here, $h_{\mu\nu}$ denotes the induced metric on the brane

In the brane world cosmology, in general, we must consider the initial and boundary conditions at the same time (Koyama and Soda, 2000; Koyama and Soda, 2002). For this reason, it is difficult to deduce the predictions such as CMB fluctuations. To circumvent this situation, we solve the 5 dimensional bulk dynamics firstly, and obtain the 4-dimensional effective theory. To carry out this program, we must adopt the low energy approximation. The procedure we performed will be referred to as holographic projection in this paper (Kanno and Soda, 2002a; Kanno and Soda, 2002b). The point is that the low energy approximation is useful because $\rho \leq (1\text{TeV})^4(1\text{cm}/l)^2$ is not so low!

2. Holographic Projection

Let us start with a review of the geometrical approach (Shiromizu, Maeda and Sasaki, 2000). In the Gaussian normal coordinate system: $ds^2 = dy^2 + g_{\mu\nu}(y, x^\mu)dx^\mu dx^\nu$, we have

$$\begin{aligned} G^{(5)}_{\mu\nu} \ &= \ G^{(4)}_{\mu\nu} + K_{\mu\nu,y} - g_{\mu\nu}K_{,y} - KK_{\mu\nu} + 2K_{\mu\lambda}K^{\lambda}_{\nu} \\ &\quad + \frac{1}{2}g_{\mu\nu}\left(K^2 + K^{\alpha}_{\beta}K^{\beta}_{\alpha}\right) = \frac{6}{l^2}g_{\mu\nu} \ , \end{aligned} \tag{2}$$

where $K_{\mu\nu} = -g_{\mu\nu,y}/2$. Taking into account the Z_2 symmetry, we also obtain the junction condition $[K^{\mu}_{\nu} - \delta^{\mu}_{\nu}K]\,|_{y=0} \ = \ \kappa^2/2\,(-\sigma\delta^{\mu}_{\nu} + T^{\mu}_{\nu})$. Here, $T_{\mu\nu}$ represents the energy-momentum tensor of the matter. Evaluating Eq.(2) at the brane and substituting the junction condition into it, we have the "effective" equations of motion

$$\begin{aligned} G^{(4)}_{\mu\nu} \ &= \ 8\pi G_N T_{\mu\nu} + \kappa^4\pi_{\mu\nu} - E_{\mu\nu} \qquad (3) \\ &\pi_{\mu\nu} = -\frac{1}{4}T^{\lambda}_{\mu}T_{\lambda\nu} + \frac{1}{12}TT_{\mu\nu} + \frac{1}{8}g_{\mu\nu}\left(T^{\alpha\beta}T_{\alpha\beta} - \frac{1}{3}T^2\right) \\ &E_{\mu\nu} = C_{y\mu y\nu}|_{y=0} \ , \end{aligned}$$

where $C_{y\mu y\nu}$ is the Weyl tensor and $8\pi G_N = \kappa^2/l$. We also assumed $\kappa^2\sigma = 6/l$.

Notice that the above geometrical projection is not the closed system. Our aim is to get the closed system of the equations. The above Eq.(3) can be transformed into a more convenient form for this purpose. In fact, the equation $\nabla^\mu E_{\mu\nu} \ = \ \kappa^4\nabla^\mu\pi_{\mu\nu}$ derived from the Bianchi identity can be integrated to

$$E_{\mu\nu} = \kappa^4\pi_{\mu\nu} - l^2\chi_{\mu\nu} - t_{\mu\nu} \tag{4}$$

where $\nabla^\mu\chi_{\mu\nu} = 0$ and $\nabla^\mu t_{\mu\nu} = 0$. Here, we have divided the integration constant into the nonlocal part $\chi_{\mu\nu}$ and the local part $t_{\mu\nu}$. The explicit formula for $\chi_{\mu\nu}$ and $t_{\mu\nu}$ can be obtained by resorting to the low energy approximation.

The low energy approximation can be reformulated as the gradient expansion as can be seen from the estimate: $\rho/\sigma \sim l^2R \ll 1$. In the gradient expansion, we can expand $\chi_{\mu\nu}$ and $t_{\mu\nu}$ as

$$\chi_{\mu\nu} \ = \ \underbrace{\chi^{(1)}_{\mu\nu}}_{\mathcal{O}(l^2R)} + \underbrace{\chi^{(2)}_{\mu\nu}}_{\mathcal{O}(l^4R^2)} + \cdots \tag{5}$$

$$t_{\mu\nu} \ = \ \underbrace{t^{(2)}_{\mu\nu}}_{\mathcal{O}(l^4R^2)} + \underbrace{t^{(3)}_{\mu\nu}}_{\mathcal{O}(l^6R^3)} + \cdots \ , \tag{6}$$

where $t_{\mu\nu}$ can be obtained from $h_{\mu\nu}$ and its derivatives. The property $E^{\mu}{}_{\mu} = 0$ leads the following relations

$$\chi^{(1)\mu}{}_{\mu} = 0 \tag{7}$$
$$l^2 \chi^{(2)\mu}{}_{\mu} = \kappa^4 \pi^{\mu}{}_{\mu} - t^{(2)\mu}_{\mu} \ . \tag{8}$$

Having obtained $\chi_{\mu\nu}$ and $t_{\mu\nu}$, we finally get the holographic projection

$$G^{(4)}_{\mu\nu} = 8\pi G_N T_{\mu\nu} + l^2 \chi_{\mu\nu} + t_{\mu\nu} \ . \tag{9}$$

It should be noted that the bulk metric can be reconstructed perturbatively from the data obtained by solving Eq.(9). That is why we use the terminology “holography” for our projection method.

3. Single brane model (RS2): AdS/CFT Correspondence

The nonlocal tensor $\chi_{\mu\nu}$ must be related to the boundary conditions in the bulk. The natural choice is asymptotically AdS boundary condition. For this boundary condition, $\chi^{(1)}_{\mu\nu} = 0$. Hence, Einstein theory is recovered at the leading order!

At $\mathcal{O}(l^4 R^2)$ order, one can take $t^{(2)\mu}_{\mu} = 0$. Indeed, the gradient expansion method gives

$$\begin{aligned} \alpha t^{(2)\mu}{}_{\nu} = & \ R^{\mu}{}_{\alpha} R^{\alpha}{}_{\nu} - \frac{1}{3} R R^{\mu}{}_{\nu} - \frac{1}{4} \delta^{\mu}_{\nu} (R^{\alpha}{}_{\beta} R^{\beta}{}_{\alpha} - \frac{1}{3} R^2) \\ & - \frac{1}{2} \left(R^{\alpha\mu}{}_{|\nu\alpha} + R^{\alpha}{}_{\nu}{}^{|\mu}{}_{|\alpha} - \frac{2}{3} R^{|\mu}{}_{|\nu} - \Box R^{\mu}{}_{\nu} + \frac{1}{6} \delta^{\mu}_{\nu} \Box R \right) \end{aligned} \tag{10}$$

which is transverse and traceless, $t^{(2)\mu}{}_{\nu|\mu} = 0$, $t^{(2)\mu}_{\mu} = 0$. Moreover, using Eq.(8) and the lowest order equation $T^{\mu}_{\nu} \approx l/\kappa^2 (R^{\mu}{}_{\nu} - \frac{1}{2} \delta^{\mu}_{\nu} R)$, we have

$$\chi^{(2)\mu}{}_{\mu} = \frac{1}{4} \left(R^{\alpha}{}_{\beta} R^{\beta}{}_{\alpha} - \frac{1}{3} R^2 \right) \ . \tag{11}$$

Utilizing the AdS/CFT correspondence, we can identify

$$\chi^{(2)}_{\mu\nu} = \frac{\kappa^2}{l^3} < T^{\rm CFT}_{\mu\nu} > \ . \tag{12}$$

Thus, we obtain the holographic effective equations of motion

$$G^{(4)}_{\mu\nu} = 8\pi G_N T_{\mu\nu} + 8\pi G_N < T^{\rm CFT}_{\mu\nu} > + \alpha t^{(2)\mu}{}_{\nu} \ . \tag{13}$$

Now we can consider the cosmology. The background spacetime is nothing but the ordinary one since the correction does not affect the

isotropic homogeneous background. It is the cosmological perturbations that we are interested in. From the renormalized action for the CFT, $S^{\rm CFT}$, we have

$$< T^{\rm CFT}_{\mu\nu} > = - < \frac{2}{\sqrt{-g}} \frac{\delta S^{\rm CFT}}{\delta g^{\mu\nu}} > \tag{14}$$

and

$$< T^{\rm CFT}_{\mu\nu}(x) T^{{\rm CFT}\lambda\rho}(y) > = - \frac{2}{\sqrt{-g}} \frac{\delta < T^{\rm CFT}_{\mu\nu} >}{\delta g^{\lambda\rho}} \ . \tag{15}$$

Hence, the perturbed effective Einstein equations are obtained as

$$\begin{aligned} \delta G_{\mu\nu} &= 8\pi G \delta T_{\mu\nu} \\ &\quad - \frac{1}{2} \int d^4 y \sqrt{-g(y)} < T^{\rm CFT}_{\mu\nu}(x) T^{{\rm CFT}\lambda\rho}(y) > \delta g_{\lambda\rho} + \alpha \delta t^{(2)}_{\mu\nu} \ . \end{aligned} \tag{16}$$

This is the integro-differential equation. Notice that $< T^{\rm CFT}_{\mu\nu}(x) T^{{\rm CFT}\lambda\rho}(y) >$ are given kernel, once the CFT is specified. Moreover, α is a parameter determined by the initial conditions for the bulk geometry. Therefore, in principle, we can obtain the time evolution of the cosmological perturbations by solving Eq.(16).

4. Two-brane model (RS1): Radion

We consider the two-brane system in this section. At the lowest order of the low energy approximation, we have the background metric

$$ds^2 = e^{2\phi(x)} dy^2 + \exp[-\frac{2}{l} e^{\phi(x)} y] h_{\mu\nu}(x) dx^\mu dx^\nu \ , \tag{17}$$

where the radion field ϕ is related to the distance between two branes as $d(x) = e^{\phi(x)} l$.

At the next order, the holographic projection gives

$$G^{(4)}_{\mu\nu}(h_{\mu\nu}) = \frac{\kappa^2}{l} T^A_{\mu\nu} + l^2 \chi^{(1)}_{\mu\nu} \tag{18}$$

$$G^{(4)}_{\mu\nu}(f_{\mu\nu}) = -\frac{\kappa^2}{l} T^B_{\mu\nu} + l^2 \chi^{(1)}_{\mu\nu} \exp[4 e^{\phi(x)}] \ , \tag{19}$$

where $f_{\mu\nu}$ is the induced metric on the B-brane. The extra factor attached to the last term in Eq.(19) can be deduced from the 5-dimensional equations. It should be noted that $h_{\mu\nu}$ and $f_{\mu\nu}$ are not independent but related as $f_{\mu\nu} = \exp[-2e^{\phi(x)}] h_{\mu\nu}$ at this order. Therefore, we can eliminate $\chi_{\mu\nu}$ from Eqs.(18) and (19), and obtain the quasi-scalar-tensor

gravity. Indeed, defining a new field $\Psi = 1 - \exp[-2e^{\phi(x)}]$, we find

$$\begin{aligned} G^{\mu}{}_{\nu}(h) &= \frac{\kappa^2}{l\Psi} T^{A\mu}{}_{\nu} + \frac{\kappa^2(1-\Psi)^2}{l\Psi} T^{B\mu}{}_{\nu} + \frac{1}{\Psi}\left(\Psi^{|\mu}{}_{|\nu} - \delta^{\mu}_{\nu}\Psi^{|\alpha}{}_{|\alpha}\right) \\ &\quad + \frac{\omega(\Psi)}{\Psi^2}\left(\Psi^{|\mu}\Psi_{|\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\Psi^{|\alpha}\Psi_{|\alpha}\right) , \end{aligned} \tag{20}$$

$$\Box\Psi = \frac{\kappa^2}{l}\frac{T^A + (1-\Psi)T^B}{2\omega(\Psi)+3} - \frac{1}{2\omega(\Psi)+3}\frac{d\omega(\Psi)}{d\Psi}\Psi^{|\mu}\Psi_{|\mu} , \tag{21}$$

where the coupling function $\omega(\Psi)$ takes the form: $\omega(\Psi) = 3\Psi/2(1-\Psi)$. This action can be a starting point of various applications such as the inflation in the two-brane system.

Importantly, we can also deduce the formula for Weyl fluid

$$\begin{aligned} \frac{l^3}{2}\chi^{(1)\mu}{}_{\nu} &= -\frac{\kappa^2(1-\Psi)}{2\Psi}\left(T^{A\mu}{}_{\nu} + (1-\Psi)T^{B\mu}{}_{\nu}\right) \\ &\quad -\frac{l}{2\Psi}\left[\left(\Psi^{|\mu}{}_{|\nu} - \delta^{\mu}_{\nu}\Psi^{|\alpha}{}_{|\alpha}\right) + \frac{\omega(\Psi)}{\Psi}\left(\Psi^{|\mu}\Psi_{|\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\Psi^{|\alpha}\Psi_{|\alpha}\right)\right] . \end{aligned} \tag{22}$$

The explicit form for the Weyl fluid is now known provided the effective equations (20) and (21) are solved.

5. Effect on CMB fluctuations

Let us briefly touch on a possible effect of bulk gravitational waves on CMB fluctuations. Taking the perturbed 4-dimensional metric as

$$ds^2 = -(1+2\phi)dt^2 + (1+2\psi)\delta_{ij}dx^i dx^j , \tag{23}$$

and defining the gauge invariant curvature perturbation

$$\zeta = \psi + \frac{\delta\rho}{3(\rho+p)} , \tag{24}$$

we have the Sachs-Wolfe formula for the CMB temperature anisotropy (Langlois et al., 2001)

$$\frac{\delta T}{T} = \zeta + \psi - \phi + \int d\eta \frac{\partial}{\partial\eta}(\psi - \phi) . \tag{25}$$

In order to give a precise value, we must know the anisotropic stress due to the Weyl fluid. For two brane system, the Weyl fluid is represented by the radion field as in Eq.(22). Hence, the formula (25) reduces to

$$\begin{aligned} \frac{\delta T}{T}|_{\rm SW} &= \frac{\delta T}{T}|_{4D} - \frac{8}{3}\frac{\rho_r}{\rho_d}S_{\rm Weyl} \\ &\quad + \frac{1}{\Psi_0}\left(\delta\Psi^{|i}_{|j} - \frac{1}{3}\delta\Psi^{|k}_{|k}\right) - \frac{2}{a^{5/2}}\int \frac{da}{\Psi_0}a^{3/2}\left(\delta\Psi^{|i}_{|j} - \frac{1}{3}\delta\Psi^{|k}_{|k}\right) . \end{aligned} \tag{26}$$

Here, the radion dynamics is important to predict the observational consequences. As we have the effective theory already, the evolution of the radion field can be calculated easily. Then, the remaining issue is to determine the initial conditions. The initial value for $\delta\Psi$ should be determined by calculating quantum fluctuations in the inflationary period. This is now under investigation.

6. Conclusion

We revealed the holographic aspects of the brane world cosmology. In the case of the single brane model, imposing the asymptotically AdS boundary condition, we obtained the Einstein equations with CFT and higher curvature corrections. As to the two-brane model, it is shown that Einstein equations with the nonlocal Weyl fluid are converted to the quasi-scalar-tensor gravity. There, the radion field played an important role. In particular, the Weyl fluid is explicitly written down using the radion field. Consequently, we succeeded to write down the formula for CMB anisotropy $\delta T/T$ in a closed form. The precise predictions will be published in the future.

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